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Numerical method for designing approximate cloaks with arbitrary shapes

Hua Ma,[1,*] Shaobo Qu,[1] Zhuo Xu,[2] and Jiafu Wang[1]
[1]*The College of Science, Air Force University of Engineering, Xi'an 710051, People's Republic of China*
[2]*Electronic Materials Research Laboratory, Key Laboratory of Ministry of Education, Xi'an Jiaotong University, Xi'an 710049, People's Republic of China*

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The design of perfect electromagnetic cloaks is usually based on the coordinate transformation method. However, for cloaks with arbitrary shapes, it is difficult to express analytically such coordinate transformation. Thus, we have to resort to numerical methods to design approximate cloaks. In this paper, we propose a numerical method of designing cloaks with arbitrary shapes. According to this method, spatial compressions are treated as homeomorphous mappings and described in numerical forms. The corresponding coordinate transformations are therefore expressed numerically. As an application of this method, we deduced the discrete constitutive parameters of a two-dimensional polygon cloak, and verified the results by numerical simulations. This method can be also used in cases with acoustic waves and matter waves.

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The coordinate transformation [1] and the optical conformal map methods [2,3] were adopted in the design of cylindrical cloaks [4,5] and were later employed in designing transformation media [6–8]. Then, those concepts were extended to cases with acoustic waves and matter waves [9–13]. Recently, the design of cloaks with different shapes was an increasingly interesting issue, but only cloaks with some regular geometrical shapes [14,15] were studied. The latest literature [16,17] has presented some analytical results in designing arbitrarily shaped cloaks, which opens up new possibilities for designing complicated cloaks. However, practically, shapes of cloaked bodies are mostly irregular and complicated, so it is difficult to analytically express the corresponding coordinate transformations. Furthermore, straight spatial compressions are mostly adopted by present theoretical analyses. They can only be used in the simple case where there is at least one inner point within the body volume from which all points on the outer surface can be reached along straight lines. But in fact, such a requirement cannot always be satisfied, for in some complicated cases, there may be no such inner point at all in the body volume. In such cases, the straight spatial compressions are not valid any more, so contorted spatial compressions have to be used. Unfortunately, it is usually very difficult to get the analytical forms of contorted spatial compressions, so when the complicated cases are considered, we have to develop numerical methods.

According to the coordinate transformation method, the procedures of designing cloaks are equivalent to spatial compressions determined by the shapes of the cloaked bodies. Thus, the bodies should first be mathematically described. However, such descriptions are usually difficult when the shapes of the bodies are complicated and irregular. To solve this problem, in another paper we proposed an approximation approach, which treats a cloaked body as a polyhedron [or a polygon in two-dimensional (2D) spaces], and describes the spatial compression geometrically using position surfaces (PSs) [or position contours (PCs) in 2D spaces] and tracing lines (TLs). We also discussed straight spatial compressions and obtained an approximate analytical expression of a polygon cloak. However, solutions to contorted spatial compressions have not yet been treated analytically. In this paper, we adopt the above approximation approach and propose a numerical solution to all forms of spatial compressions. This numerical solution can be used to approximately design arbitrary cloaks with more complicated shapes for whatever straight or contorted spatial compressions are adopted. In the following, we will introduce this method and, for the sake of simplicity, we will present examples in 2D spaces.

Similar to equal potential surfaces, PSs are such surfaces that all the points that have "equal positions" are on PSs. By assigning all the points that fall into a studied region to a series of PSs and by considering the shape changes of these PSs, spatial compressions are described. When the spatial compression is performed, the points on PSs will have traces in the space and TLs are formed by these traces. Obviously, TLs and spatial compressions are correspondingly correlated: straight TLs corresponding to straight spatial compressions and contorted TLs to contorted spatial compressions. In fact, a spatial compression can be treated as a homeomorphous topological mapping (HTM) between two similar topology structures. This determines the rule of drawing PSs and TLs. That is to say, PSs and TLs should be drawn by such a rule that there is no intercrossing between all the PSs whereas for each PS, it must intercross with every TL only once and for all the TLs, there is no intercrossing between them but they must intercross with each PS once.

We sketch PCs and TLs in 2D spaces in Fig. 1, where an arbitrarily shaped body is approximated by a polygon and is divided into many triangles (see the left panel in Fig. 1). Notice that several triangles are contorted [for instance, the gray region in Fig. 1(a)] due to the complicated shape of the cloaked body. In such a body, there is no inner point from which all the points on the outer surface can be reached along straight lines. The body thus has polygonal PCs and contorted TLs in some regions. We thus draw the conclusion that in the simple cases there is at least one inner point in the

*mahuar@163.com

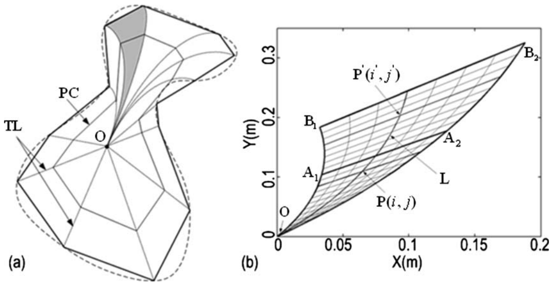

FIG. 1. Sketches of PCs and TLs in 2D spaces (a) an arbitrarily shaped body (dashed lines) is approximated by a polygon (solid lines, bold) and is divided into many triangles by TLs; (b) a contorted triangle [gray region in (a)] is zoomed in and the curves that converge at O are TLs while the straight lines that intercross with TLs are PCs.

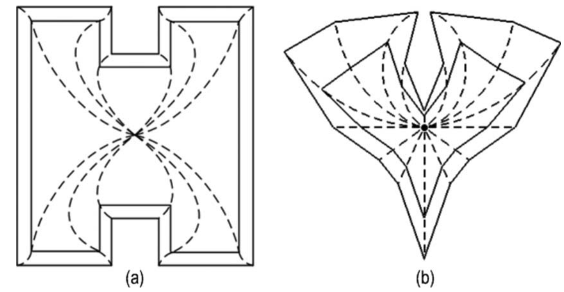

FIG. 2. In the cloaked body, if there is no inner point from which all the points on the outer surface can be reached along straight lines, the frame TLs (dashed lines) are (a) contorted (for H-shaped cloaks) or (b) alternately straight and contorted in various regions (for bird-shaped cloaks).

cloaked volume from which straight lines can be drawn to every point on the outer surface, the triangles divided by TLs are straight, and the spatial compression is also straight. However, in complicated cases where there is no such inner point, the TLs are contorted and the contorted spatial compression has to be used. As a result, when simple cases are considered, coordinate transformations with respect to straight spatial compressions can be expressed analytically, whereas when complicated cases are considered, coordinate transformations with respect to contorted spatial compressions cannot be expressed in analytical forms. The solution proposed in this paper will be a generalized numerical method for the two kinds of cases and will be introduced in the following.

We illustrate the method in 2D spaces. Here, a polygon, which is used to approximate to an arbitrarily shaped body and has polygonal PCs, is considered. First, we discuss the drawing of TLs. As shown in Fig. 1(b), in the contorted triangle B_1OB_2, the curves that converge at the point O are TLs, and the straight lines that intercross with these TLs are PCs. We call the borders OB_1 and OB_2 frame TLs which plays a key role in describing the spatial compression. The frame TLs can be drawn according to their position data obtained through numerical methods. Here, we offer a simple algorithm of describing the frame TLs. Suppose the original point is O with coordinates $x_0=0$ and $y_0=0$ and j is used to number a point on the considered frame TL. Then, the discrete function of the frame TL is

$$x_j = x_{j-1} + d_j \cos \theta_j, \quad y_j = y_{j-1} + d_j \sin \theta_j, \tag{1}$$

where θ_j describes the frame TL curvature and d_j the discrete scale. Obviously, the TLs described by Eq. (1) may be straight, contorted, or alternately straight and contorted in various regions, so Eq. (1) is the general form of numerically describing frame TLs. Note that, under some conditions, this equation can be simplified by supposing d_j to be a constant and $\theta_j=\theta_0+(j-1)\Delta\theta$.

As a result, based on Eq. (1), the arc length of the considered frame TL from the original point O to point j is

$$l_j = \sum_{k=1}^{j} d_k. \tag{2}$$

After drawing two frame TLs of a contorted triangle, the PCs and other TLs can be created correspondingly by numerical methods. One of the simplest methods is as follows. As shown in the contorted triangle B_1OB_2 in Fig. 1(b), we can divide the arc lengths of OB_1 and OB_2 into N (integer, >2) equal or unequal portions according to Eq. (2), number the end points of these portions by j, connect the two points identified by the same j, respectively, on OB_1 and OB_2, and regard the connecting lines as PCs. Then, we divide each PC into M (integer, >2) equal portions, number the end points of these portions by i, connect the points identified by the same i between all PCs, and finally create a series of curves and treat these curves as TLs. Correspondingly, as shown in Fig. 1(b), the point at which TL i and PC j intercross with each other can be identified by (i,j) and the position $(x_{i,j}, y_{i,j})$ can also be obtained.

To illustrate the above method of plotting TLs and PCs, we present our instances in Fig. 2 in which an H-shaped cloak and a bird-shaped cloak are sketched, respectively. As we all know, for bodies that have the above two kinds of shapes, there is no inner point from which all the points on the outer surfaces can be reached along straight lines, so the frame TLs must be contorted. In Fig. 2, we plot the frame TLs of the H-shaped cloak using Eq. (1) and plot the frame TLs of the bird-shaped cloak using the simplified form of Eq. (1) by supposing d_j to be a constant and $\theta_j=\theta_0+(j-1)\Delta\theta$. Figure 2 shows that Eq. (1) is an effective approach to create the frame TLs for complicated shapes. Note that the approaches of creating frame TLs are not unique, and that any approach that obeys the principle of HTM can be used.

Next, we need to define a spatial compression. Generally, various methods can be adopted to create the spatial compression in performing perfect invisibility. However, linear compressions have the simplest forms and are widely used, so we adopt the following transformation:

$$l' = l_a + \frac{l}{l_b}(l_b - l_a), \tag{3}$$

where l_a and l_b are the arc lengths of a frame TL from the original point O to the point where the TL intercrosses the

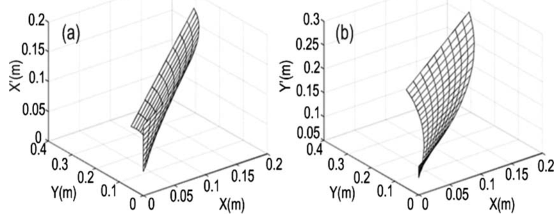

FIG. 3. Sketches of the surfaces: (a) $x'=x'(x,y)$ and (b) $y'=y'(x,y)$.

inner and outer shells of the cloak, l and l' the arc lengths of a frame TL from the original point O to the considered point j and to the mapping point j', respectively. Obviously, Eq. (3) determines the spatial compression and therefore creates the mapping between the PCs j and j'. As a result, the mapping between the points (i,j) and (i,j') is created uniquely. As shown in Fig. 1(b), for instance, points $\mathrm{P}(i,j)$ and $\mathrm{P}'(i',j')$ are both in TL and are mapped to each other, where $i'=i$. In this way, the region $\mathrm{B_1OB_2}$ is compressed into the shell region $\mathrm{A_1A_2B_2B_1}$ numerically.

Once the mapping between the points (i,j) and (i',j') is created, we can now perform the coordinate transformation to calculate the constitutive parameters. Suppose that the coordinates of a considered point is (x,y) and that its mapping position is (x',y'). Then, the Jacobian transformation matrix A between these two coordinates is thus defined as follows:

$$A_{11}=\frac{\partial x'}{\partial x},\quad A_{12}=\frac{\partial x'}{\partial y},$$

$$A_{21}=\frac{\partial y'}{\partial x},\quad A_{22}=\frac{\partial y'}{\partial y}. \tag{4}$$

The above formula requires $x'=x'(x,y)$ and $y'=y'(x,y)$ to be explicitly expressed, but it is usually very difficult for a complicated cloak with arbitrary shapes, so, we have to develop some methods to offer the numerical results. In Eq. (4), we notice that (A_{11},A_{12}) and (A_{21},A_{22}) are the gradients of variables x' and y', respectively. This indicates that if we numerically plot the surfaces $x'=x'(x,y)$ and $y'=y'(x,y)$, we can then calculate the gradients of x' and y'.

As shown in Fig. 3, the point $(x_{i,j},y_{i,j},x'_{i,j})$ is on the surface $x'=x'(x,y)$ and the point $(x_{i,j},y_{i,j},y'_{i,j})$ on the surface $y'=y'(x,y)$, respectively. By linear approximation, we obtain the numerical results of A as follows:

$$A_{11}(i,j)=\frac{(y_{i+1,j}-y_{i,j})(x'_{i,j+1}-x'_{i,j})-(y_{i,j+1}-y_{i,j})(x'_{i+1,j}-x'_{i,j})}{(x_{i+1,j}-x_{i,j})(y_{i,j+1}-y_{i,j})-(y_{i+1,j}-y_{i,j})(x_{i,j+1}-x_{i,j})},$$

$$A_{12}(i,j)=\frac{(x_{i,j+1}-x_{i,j})(x'_{i+1,j}-x'_{i,j})-(x_{i+1,j}-x_{i,j})(x'_{i,j+1}-x'_{i,j})}{(x_{i+1,j}-x_{i,j})(y_{i,j+1}-y_{i,j})-(y_{i+1,j}-y_{i,j})(x_{i,j+1}-x_{i,j})},$$

$$A_{21}(i,j)=\frac{(y_{i+1,j}-y_{i,j})(y'_{i,j+1}-y'_{i,j})-(y_{i,j+1}-y_{i,j})(y'_{i+1,j}-y'_{i,j})}{(x_{i+1,j}-x_{i,j})(y_{i,j+1}-y_{i,j})-(y_{i+1,j}-y_{i,j})(x_{i,j+1}-x_{i,j})},$$

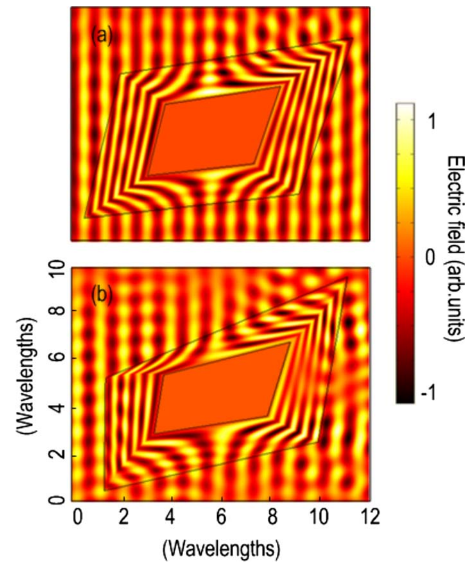

FIG. 4. (Color online) In the case electromagnetic waves are incident onto a quadrangle cylinder cloak, the electric-field distributions for different spatial compressions: (a) straight spatial compressions and (b) contorted spatial compressions.

$$A_{22}(i,j)=\frac{(x_{i,j+1}-x_{i,j})(y'_{i+1,j}-y'_{i,j})-(x_{i+1,j}-x_{i,j})(y'_{i,j+1}-y'_{i,j})}{(x_{i+1,j}-x_{i,j})(y_{i,j+1}-y_{i,j})-(y_{i+1,j}-y_{i,j})(x_{i,j+1}-x_{i,j})}. \tag{5}$$

Finally, by numerically creating the corresponding HTM between the original points and their new positions, we obtain the Jacobian transformation matrix A that will be used in calculating the constitutive parameters of a considered cloak according to the method stated in Ref. [6]. Note that the above procedures are of generalized forms, and the numerical results may not be unique since the spatial compression and the algorithm of drawing frame TLs are optional. As a result, when different spatial compressions and algorithms are adopted, the field distributions in the cloaking shell changes correspondingly. However, such differences have no impact on the outer fields of the cloak, and thus cloaking performances are the same (see Fig. 4).

To verify the method introduced above, we have finished the full-wave finite-element simulations by using the COMSOL multiphysics solver. In each simulation, we create several data files. In one of these files, stored is the information about the frame TLs, and in the others, stored is the data of constitutive parameters. Generally, there are much data in these files, which requires the computer to be of high capacity. In addition, the computation domain should be configured carefully. In our simulations, we set the inner shell of the cloak to be perfect electric conducting (PEC) and the boundaries that are parallel to the wave propagation to be perfect magnetic conducting (PMC).

The first simulation gives a comparison between the straight and the contorted spatial compressions. As seen in Fig. 4, we design a quadrangle cylinder cloak by using a straight spatial compression and a contorted spatial compression, respectively. The figure clearly shows that when the straight spatial compression is adopted [see Fig. 4(a)], the

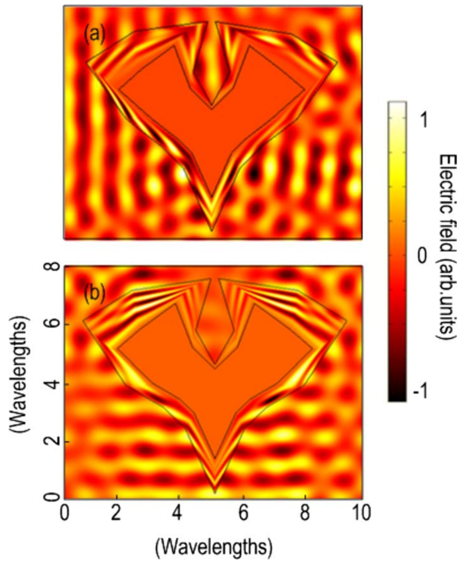

FIG. 5. (Color online) The electric-field distributions for the case electromagnetic waves are incident onto a bird-shaped cloak (a) horizontally or (b) perpendicularly.

wave vector lines in the corners are straight while when the contorted spatial compression is adopted [see Fig. 4(b)], the wave vector lines in the corners are contorted. Nonetheless, the outer field distributions are almost the same in the two cases, which therefore indicates the same cloaking performances.

The second simulation is carried out for a bird-shaped cloak whose frame TLs are sketched in Fig. 2. In the simulation, as shown in Fig. 5, the electromagnetic waves are incident onto the cloak horizontally or perpendicularly. The figure shows that, inside the cloaking shell, the phase fronts are warped, which causes power flow to bend around the inner shell. Yet, in the outer region of the cloak, the phase fronts still keep planar. The results clearly show the perfect invisibilities of the cloak and give an immediate verification for our method. However, we also notice that, as shown in Fig. 5, the scattering outside the cloak is obvious. By increasing the number of the computational elements, we reduce the scattering to a lower level. Theoretically, the scattering can be completely eliminated, but in practice, the accuracy of simulations is limited by the division method and the system configuration of the software, so the computation method and the configuration of the software can be improved to reduce the scattering.

Some other problems with this numerical method should be mentioned. First of all, the evaluation of the accuracy of the method needs to be discussed. As we know, many factors may impact the accuracy. Those factors include the discrete approaches, the number of computational elements, and the algorithm of creating PSs/PCs and TLs. Secondly, this method is actually an approximation approach, which requires us to evaluate the approximation errors produced by a polyhedron. In addition, the evaluation of the RCS of the cloaked body and the effects on the mechanical performances of a numerically designed cloak should be further studied carefully.

In conclusion, a numerical method of designing arbitrarily shaped cloaks was investigated. By adopting contorted spatial compressions and using geometrical descriptions (PSs/PCs and TLs), this method numerically creates the HTM between the original points and their mapping positions when spatial compressions are performed, and thus numerically expresses the corresponding coordinate transformations. According to this method, for an arbitrarily shaped body, whether it is simple or complicated, the cloak will be numerically designed as long as the frame TLs can be created following the principle of HTM. Although it is introduced in 2D spaces, the method can be easily extended to the three-dimensional cases. Furthermore, this method is also applicable to cases with acoustic waves and matter waves.

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